Stat 344
Life Contingencies I

Chapter 7: Policy values / Reserves

Future loss random variables

When we discussed the setting of premium levels, we often made use of future loss random variables.

- In that context, we only considered future loss random variables at issue, i.e., future loss at time 0.

We're again to make use of future loss random variables in order to study policy values.

- To do this, we'll need to consider future loss random variables at arbitrary time points after policy issue.

General Formula

Future loss at time $t =$
PV_t (future "benefits") $- PV_t$ (future "premiums")

Thus, the future loss is *only* concerned with events happening after time t — benefits and premiums occurring prior to time t do not affect this calculation.

Future loss random variables

The specific random variables we'll utilize are:

Net Future Loss at Time t Random Variable

$$L_t^n = PV_t(\text{future benefits}) - PV_t(\text{future net premiums})$$

Gross Future Loss at Time t Random Variable

$$L_t^g = [PV_t(\text{future benefits}) + PV_t(\text{future expenses})] - [PV_t(\text{future gross premiums})]$$

The notation $PV_t(X)$ denotes the present value at time t of X.

If t is an integer, then any cash flows occurring at the end of year t *will not* be included in these calculations, whereas cash flows occurring at the start of year $t+1$ *will* be included.

Net future loss random variable — Example

Consider a 40-year old that has purchased a whole life insurance policy with $\$100,000$ payable at the end of the year of death. Premiums are payable at the beginning of the year.

Using the Standard Ultimate Mortality Table with $i = 5\%$ and the Equivalence Principle to set the annual premium gives $P = 655.88$.

At issue:

$$L_0^n = 100,000\, v^{K_{40}+1} - 655.88\, \ddot{a}_{\overline{K_{40}+1}|}$$

and

$$E[L_0^n] = 0$$

Net future loss random variable — Example (continued)

Then at time 10:

$$L_{10}^{n} = 100,000\, v^{K_{50}+1} - 655.88\, \ddot{a}_{\overline{K_{50}+1|}}$$

so that

$$\begin{aligned} E[L_{10}^{n}] &= 100,000 A_{50} - 655.88\, \ddot{a}_{\overline{K_{50}+1|}} \\ &= 100,000(0.18931) - 655.88 \left(\frac{1 - 0.18931}{0.0476} \right) \\ &= \$7,760.51 \end{aligned}$$

Unlike at issue, this expected future loss is not zero; the future premiums are *not* expected to be sufficient to cover future benefits. The insurer would need to have, on average, \$7,760.51 on hand in addition to future premiums in order to cover the future benefits.

Policy values

The amount needed to cover the shortfall between future benefits and future premiums ($\$7,760.51$ in the previous example) is called the **policy value** at time t and is denoted generically by ${}_tV$.

The process of calculating policy values is known as **valuation**.

- Sometimes (especially in the U.S.) it's called **reserving**, and policy values are known as **reserves**.

The general "prospective" formula for a policy value is

Prospective Policy Value Formula

$${}_tV = EPV_t(\text{future "benefits"}) - EPV_t(\text{future "premiums"})$$

Policy value bases

The assumptions (such as mortality, expenses, interest, etc.) used in a policy value calculation form the **policy value basis**. In contrast, the assumptions used to originally calculate the premiums for the policy form the **premium basis**.

- There's no real reason to think that these two bases will be the same, and in general, they will indeed differ.

The policy value basis used will typically depend on the purpose of the valuation. Some common reasons for valulations:

1. Internal management information
2. Regulatory requirements
3. Shareholder reporting
4. Reporting required for taxation purposes

Policy values

The **gross premium policy value** at time t is the expected value (at time t) of the gross future loss random variable.

- The premiums used in the calculation are the actual gross premiums for the policy.

The **net premium policy value** at time t is the expected value (at time t) of the net future loss random variable.

- The premiums used in the calculation are the net premiums calculated by the Equivalence Principle, applied at the age of policy issue, calculated on the policy value basis.
- No expenses are taken into account.

Policy values calculation — Example 7.2

Consider a whole life policy for (50) with $\$100,000$ death benefit payable at the end of the year of death. Gross premiums of $\$1,300$ are paid annually.

1. Calculate the gross premium policy value at time 5, assuming the policy is still in force using the following basis:
   - Standard Select Survival Model (this is a 2-year select model)
   - $i = 5\%$
   - Expenses are 12.5% of each premium
2. Calculate the net premium policy value at time 5, using the same basis as above, but $i = 4\%$.

Results on next slide.

Policy values calculation — Example 7.2 (continued)

1. Writing out the gross future loss random variable:

$$L_5^g = \left[100,000\, v^{K_{55}+1} + (0.125)(1,300)\ddot{a}_{\overline{K_{55}+1|}}\right] - \left[(1,300)\ddot{a}_{\overline{K_{55}+1|}}\right]$$

   Then the gross premium policy value is

$$E[L_5^g] = {}_5V^g = 100,000\, A_{55} - (0.875)(1,300)\,\ddot{a}_{55} = 5,256.35$$

2. First calculate the net premium: $P = \dfrac{A_{[50]}}{\ddot{a}_{[50]}} = 1,321.31$

$$L_5^n = \left[100,000\, v^{K_{55}+1}\right] - \left[(1,321.31)\ddot{a}_{\overline{K_{55}+1|}}\right]$$

   Then the net premium policy value is

$$E[L_5^n] = {}_5V^n = 100,000\, A_{55} - (1,321.31)\,\ddot{a}_{55} = 6,704.75$$

Recursive formulas for policy values

We can develop some useful recursive formulas for our policy values, that is, formulas relating a policy value at time t to the policy value for the same policy value at time $t+1$.

These formulas are useful in practice because they allow us to calculate policy values without having to start from scratch every time.

The general idea for developing this type of formula is the same as for the recursive life insurance and annuity EPV formulas:

1. Write out the policy value at time t in terms of the EPV for all future policy cash flows.
2. Split the future cash flow values into those occurring in the first year and those occurring in later years.
3. Regroup the EPV for later year cash flows in terms of a policy value at time $t+1$.

Recursive formulas for policy values — Example

We'll use Example 7.2 to illustrate how to develop a recursion formula for a gross premium policy value.

- Having previously calculated ${}_5V^g$, we'll derive a formula that allows us to use this policy value to calculate ${}_6V^g$.
- We need one additional piece of information, namely $q_{55} = 0.00199$.

$$ {}_5V^g = [(100,000)A_{55} + (0.125)(1,300)\ddot{a}_{55}] - [(1,300)\ddot{a}_{55}] $$

$$ \left({}_5V^g + 1,300 - (0.125)(1,300)\right)(1+i) = (100,000)q_{55} + p_{55}\left({}_6V^g\right) $$

$$ {}_6V^g = 6,527.53 $$

Recursive formulas for policy values

Using the following generic notation:

P_t	premium payable at time t
e_t	expenses payable at time t
S_{t+1}	death benefit payable at time $t+1$ if the insured dies during the year
E_{t+1}	termination-related expenses payable at time $t+1$
i_t	annual effective interest rate in effect from time t to time $t+1$

The general recursion equation for life insurance policy values is:

Generic Policy Value Recursion Equation — Annual Case

$$({}_tV + P_t - e_t)(1 + i_t) = q_{[x]+t}\,(S_{t+1} + E_{t+1}) + p_{[x]+t}\;{}_{t+1}V$$

Other versions of policy value formulas

In addition to the prospective and recursive formulas we've seen for policy values, we can also derive various other formulas for the **net** policy value, usually by manipulating the prospective formula.

For example, consider a whole life policy with \$1 death benefit payable at the end of the year of death and annual premiums payable in advance, issued to (x). For this case,

$$ {}_tV^n = A_{x+t} - \frac{A_x}{\ddot{a}_x}\,\ddot{a}_{x+t} = \left(\frac{A_{x+t}}{\ddot{a}_{x+t}} - \frac{A_x}{\ddot{a}_x}\right)\ddot{a}_{x+t} $$

so that the policy value is the EPV of the "premium difference" payable over the remaining life of the policy.

We can also often express a net policy value entirely in terms of life insurance EPVs or annuity EPVs.

Non-anniversary policy values

Thus far we've only calculated policy values on policy anniversary dates, i.e., integral numbers of years after policy issue.

- We could also consider policy values for non-anniversary dates as well.
- The general principle is the same, though the specific calculations may get somewhat messier.

Example:

$$ {}_{3.6}V^{n} = EPV_{3.6}(\text{future benefits}) - EPV_{3.6}(\text{future premiums}) $$

Note that the policy value is *not* a monotonic function of time, so that interpolating between anniversary policy values is not expected to yield a good approximation to a non-anniversary policy value.

Policies with non-annual (discrete) cash flows

Another complication we may encounter in valuation lies in dealing with policies having non-annual cash flows.

- For example, premiums for some policies may be payable on a non-annual (e.g., quarterly, monthly) basis.

Again, the basic principle for dealing with these cases is the same, though the specific calculations may get somewhat messier.

We can also develop recursion equations for policies having (discrete) non-annual cash flows.

- These will mirror the corresponding recursions for the annual case.

Policies with continuous cash flows

Using the notation:

$\bar{P}_t$	annual rate of premium payable at time t
$\bar{e}_t$	annual rate of expenses payable at time t
S_t	death benefit payable if the insured dies at time t
E_t	termination-related expenses payable at time t
δ_t	force of interest in effect at time t

We can derive a continuous-time analog of our policy value recursion equation:

Thiele's Differential Equation

$$\frac{d}{dt}\,{}_tV = \delta_t\,{}_tV + \bar{P}_t - \bar{e}_t - \left(S_t + E_t - {}_tV\right)\mu_{[x]+t}$$

Analytical Solution of Thiele's Differential Equation — Euler's Method

We can use Thiele's Differential Equation as an approximation for a small time period h by assuming that

$$\frac{d}{dt}\,{}_tV \approx \frac{1}{h}\left({}_{t+h}V - {}_tV\right), \text{ yielding}$$

Euler's Method

$${}_{t+h}V - {}_tV = h\left(\delta_t\,{}_tV + P_t - e_t - \left(S_t + E_t - {}_tV\right)\mu_{[x]+t}\right)$$

This can be used recursively to find policy values at fractional ages.

- It's particularly helpful in cases where we know a "boundary condition"; e.g., for any endowment or insurance policy, we know the policy value as the contract approaches maturity.

Expense Reserves

We can define an **expense reserve** as the difference between the gross premium and net premium reserves:

$$ {}_tV^e = {}_tV^g - {}_tV^n, \text{ or equivalently,} $$

$$ {}_tV^e = \text{EPV of future expenses} - \text{EPV of future expense loadings} $$

Because the expenses loadings are level and expenses are typically incurred disproportionately at the beginning of the policy, the expense reserve will typically be negative at positive durations.

- This negative expense reserve is often referred to (especially in the U.S.) as the **Deferred Acquisition Cost** or **DAC**.

Expense Reserve Calculation Example

A fully discrete \$100,000 whole life policy is issued to (40). Mortality is given by the SULT and interest is 5%. Expenses are as follows:

- \$500 at issue
- Maintenance expenses of \$50 at the start of renewal years
- Termination costs of \$100
- 2% of all premiums

Calculate the net premium reserve, gross premium reserve, and expense reserve at time 5. [3475.89; 3044.87; -431.02]

$P^g = 745.83$ $\qquad E(In) = E(Out)$ $\qquad P^g \ddot{a}_{40} = 100000 A_{40} + 450 + 50\ddot{a}_{40} + 100 A_{40} + 0.02 P^g \ddot{a}_{40}$

$P^n = 655.87$

${}_5V^g = 100100 A_{45} + 50\ddot{a}_{45} + 0.02 P^g \ddot{a}_{45} - P\ddot{a}_{45}$

Modified Premium Reserves

Modified premium reserves (which are sometimes just called modified reserves) result from calculating a net premium reserve using net premiums that are not level, but follow some other specified pattern.

- The premiums are only modified for the purposes of the reserve calculation — the actual premiums paid are unchanged.
- Modified premium reserves offer some of the calculational convenience of net premium reserves, but tend to be less conservative — they're closer in value to the gross premium reserve.

Full Preliminary Term (FPT) Reserves

One particular type of modified premium reserve uses the **Full Preliminary Term (FPT)** method.

In this method, we find a single net premium (α) and renewal net annual premium (β).

- α is the EPV of the first year policy benefits, and
- The EPV of the renewal premiums is equal to the EPV of the benefits in the subsequent years.

These modified premiums are then used in the reserve calculation:

$$ {}_tV^{FPT} = \text{EPV(benefits)} - \text{EPV(modified net premiums)} $$

As a consequence of this premium pattern, ${}_1V^{FPT}$ will always be 0.

Non-forfeiture options

In some jurisdictions, when a policyholder wants to surrender (lapse) a life insurance policy that has built up a positive policy value (under a statutorily specified policy value basis), the insurer may be required to return part or all of this policy value (or perhaps the associated asset share) to the policyholder in some manner.

The possibilities available to the policyholder are known as their **non-forfeiture options**.

Some of the common non-forfeiture options are:

- Cash
- Automatic Premium Loan
- Reduced Paid-up Insurance
- Extended Term Insurance

Retrospective policy values

The basic formulas we've seen for policy values have been prospective in nature, meaning that at time t we're computing the policy value by considering what's expected to happen in the future.

We can also define a retrospective policy value by looking from time t back to the time of policy issue. The general form of a retrospective policy value is

$$\begin{aligned} &\text{retrospective policy value at time } t = \\ &\qquad \text{accumulated value at time } t \text{ of past premiums} \\ &\qquad - \text{accumulated value at time } t \text{ of past benefits and expenses,} \end{aligned}$$

where the accumulations are done under the assumptions specified in the policy value basis, with respect to mortality and interest.

Retrospective policy values

Note that the retrospective and prospective policy values are in general *not equal* unless:

- The premium was calculated by the equivalence principle and
- The premium basis is the same as the policy value basis

While the retrospective policy value is conceptually similar to the asset share, these two quantities will likely not be the same unless the insurer's actual experience was exactly the same as the assumption in the policy value basis, which is exceedingly unlikely to happen in practice.

Retrospective policy value example

Calculate ${}_2V$ for a fully discrete 10-year term policy issued to (40) both retrospectively and prospectively, using the following information:

- $q_{40+k} = 0.1 + 0.005\,k$, for $k = 0, 1, \ldots, 9$.
- $i = 8\%$
- The death benefit is \$200,000 for the first four years, \$400,000 for the next three years, and \$300,000 for the final three years.
- The net premium for the policy is \$28,327.56.

Both methods produce the same value of \$24,923.21.

In general, it's easier to compute a policy value retrospectively than prospectively when there are non-level premiums / benefits and the calculation is done at an early duration, before the changes have occurred.

(60) discrete

300,000 WL insurance policy with annual premiums

SULT 0.05

250 in first year
50 renewal years
0.08P in first
0.02P renewal years
3,000 termination exp.

$\alpha(12)\ddot{a} - \beta(12)$

526.07
514.09
[525.72]

Net Prem [5842.96]

Gross Prem [6111.64]

${}_5V^n$ [27,260.20]

${}_5V^g$ [27,017.53]

${}_5V^{FPT}$ [22,510.49]

${}_5V^e$ [-242.77]

Monthly gross premium (UDD)

L_0^g $\left[303000v^{K_{60}+1} + 200 + 50\ddot{a}_{\overline{K_{60}+1}|} + 0.06P + 0.02P\ddot{a}_{\overline{K_{60}+1}|} - P\ddot{a}_{\overline{K_{60}-1}|}\right]$

$E(L_0^g)$ 0

$Var(L_0^n)$ 4.3B

NetPremium using PPPP [6074]

$L_0^n = 300000v^{K_{60}+1} - P\ddot{a}_{\overline{K_{60}+1}|}$

$n = 1000$ $\alpha = 0.95$ $z^* = \dfrac{-nE(L_0)}{\sqrt{n\,Var(L_0)}}$

$_{10|20}q_{40}$ $\quad \frac{l_{50}-l_{70}}{l_{40}}$ $\quad {}_{10}p_{40}\,{}_{20}q_{50}$ $\quad \mu_{40}$

$\bar{A}_{40}$ $\quad \int_0^\infty e^{-\delta t} e^{-\mu t} \mu \, dt$ $\quad \frac{\mu}{\delta+\mu}$

${}_{10}p_{40}$ with const. μ

$e^{-10\mu}$

$\bar{a}_{40}$ $\quad \int_0^\infty e^{-\delta t} e^{-\mu t} dt$ $\quad \frac{1}{\delta+\mu}$

$A^{1}_{40:\overline{10}|} = A_{40:\overline{10}|} - {}_{10}E_{40}$

$A_{40} - {}_{10}E_{40} A_{50}$

$\ddot{a}_{\overline{40:10}|} = \ddot{a}_{\overline{10}|} + {}_{10}E_{40}\,\ddot{a}_{50}$

$A_{\overline{40:10}|} =$

$A^{(12)}_{40} = \frac{i}{i^{(12)}} A_{40}$

$S_{40}(20) = {}_{20}p_{40}$